A NEARLY-PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. By utilizing a combination of properties of the consequent mapping with the Brouwer's fixed point theorem we obtain existence results for the nearly-periodic boundary value problem

$$x'' = f(t, x, x'), \ \ t \in [0, 1]$$

$$x(1) = Q_0^{-1}x(0), x'(1) = Q_1^{-1}x'(0),$$

where Q_0, Q_1 are complex valued nonsingular matrices.

1. INTRODUCTION

Let $\mathbb{C}^n$ denote the n-dimensional complex Euclidean linear space and let I be the interval $I := [0, 1]$. Let Ω be a convex, open subset of the product space $\mathbb{C}^n \times \mathbb{C}^n$ and let $f : I \times \Omega \to \mathbb{C}^n$ be a continuous function. In this paper we provide sufficient conditions for the existence of a (complex valued) solution x of the vector differential equation

$$x'' = f(t, x, x'), \ \ t \in I \tag{1.1}$$

satisfying the conditions

$$x(1) = Q_0^{-1}x(0), \ \ x'(1) = Q_1^{-1}x'(0), \tag{1.2}$$

where Q_0, Q_1 are nonsingular $n \times n$ complex valued matrices. The problem under investigation is inspired by the periodic problem (in the real case) concerning (1.1), for which the literature is volunimous, as well as by those problems presented in [2, 4]. In [2] the existence of a Sturm-Liouville boundary value problem is investigated, by transforming it into the equivalent form $Lx = Gx$ and then applying the Leray-Schauder's continuation theorem. Also we would like to refer to [5, p. 338], where by using the Wazewski's method it was shown that, if in (1.1) the function f satisfies the well known Hartman's condition for all $t \geq 0$, x and $y \neq 0$, then there is a time $t_0 > 0$ such that $x(t).x(t)$ is nonincreasing on $t \geq t_0$, where x is the solution of (the real version of) equation (1.1). For a two-point boundary value problem concerning a more general differential equation in a Hilbert space discussed by the authors in [7] the Schauder' s fixed point theorem is used. Notice that in [4], the existence

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of a solution x of the problem is investigated, where the nonsingular $n \times n$-square matrices Q_0 and Q_1 satisfy the inequalities

$$x.Q_0Q_1^{-1}y \leq 0 \text{ and } x.(Q_0 + Q_1^{-1})y \leq 0, \tag{1.3}$$

for all vectors $x, y \in \mathbb{R}^n$ with $x.y \leq 0$ and the matrix Q_0 is orthogonal. (The dot denotes the inner product in the real Euclidean space.)

The literature shows a great number of papers refered to both the scalar and the vector case for the problem (1.1), (1.2), see, e.g., [1, 9, 10, 11] and the references therein. In [3] Erbe by using a technique, which involves a direct application of properties of Leray-Schauder degree, instead of (1.3), he used the following condition:

$$\text{There is a } \mu > 0 \text{ such that } Q_1 = \mu Q_0.$$

A more general situation of the problem is discussed by the authors in [8]. In this paper we do use of the Hartman's condition and give information on the existence of solutions by combining properties of the consequent mapping with the Brouwer's fixed point theorem. Motivated from Erbe's technique, instead of the orthogonality condition on Q_0, we assume that the matrices Q_0, Q_1 satisfy the relation

$$\|Q_1\| \leq \|Q_0\| = \|Q_0^{-1}\| = 1, \tag{1.5}$$

where $\|.\|$ stands for the norm in the $n \times n$ complex matrix space congruent to the euclidean norm of the complex n-dimensional space $\mathbb{C}^n$, the norm which equals to the greatest absolute value of its eigenvalues.

2. Preliminaries

Let J be a fixed interval of the real line such that $I \subset J$. Consider equation (1.1) associated with the initial conditions

$$(\tau, x(\tau), x'(\tau)) =: (\tau, \xi, \eta) =: P \in I \times \Omega, \tag{2.1}$$

where the function $f : J \times \Omega \to \mathbb{C}^n$ is continuous. Let $X(P)$ be the family of all solutions of (1.1), (2.1). If x is such a solution, we shall write I_x to denote the connected set of all existence times of x lying in I and such that $0 \in I_x$. We let $D := J \times \Omega$ and consider this set as a subset of the euclidean space $\mathbb{R} \times \mathbb{C}^n$. Take a subset W of D such that both the sets $int(W)$ and $D - cl(W)$ are nonempty. (Here $int(W)$ denotes the interior and $cl(W)$ the closure of the set W.) Later on the set W will be completely definite.

Next we recall some classical definitions. Given a $\tau \in (0, 1]$, a point $P := (\tau, \xi, \eta)$ of the boundary of W (if such exists) is *a point of egress*, if, given any $x \in X(P)$, there is an $\epsilon > 0$ such that

$$\{(t, x(t), x'(t)) : t \in (\tau - \epsilon, \tau)\} \subset int(W).$$

Also, if $\tau < 1$, then P is *a strict egress point*, if, given any $x \in X(P)$, there is an $\epsilon > 0$ such that

$$\{(t, x(t), x'(t)) : t \in (\tau, \tau + \epsilon)\} \subset D - cl(W).$$

(See, e.g., [6].) We denote by W^e and W^{se}, respectively, the sets of egress and strict egress points of W.

A point P of the boundary of W is *a consequent point* of $P_0 := (\tau_0, \xi_0, \eta_0)$, if there is a solution passing from both these points and such that

$$\{(t, x(t), x'(t)) : t \in (\tau_0, \tau)\} \subset int(W).$$

The set of all consequent points of P_0 will be denoted by $C(P_0)$, while the so defined (set-valued) mapping

$$\mathrm{C} : N_c(W) \to W^e$$

is *the consequent mapping.* Here the symbol $N_c(W)$ stands for the set of all points of W whose the sets of the consequent points are nonempty.

Given a time $\tau \in (0, 1]$ we say that a point $P := (\tau, \xi, \eta)$ of the boundary of W is *a point of ingress* of W, if given any solution $x \in X(P)$ there is an $\epsilon > 0$ such that

$$\{(t, x(t), x'(t)) : t \in (\tau - \epsilon, \tau)\} \subset D - cl(W).$$

Also, in case $\tau < 1$, the point P is *a strict ingress point*, if given any $x \in X(P)$, there is an $\epsilon > 0$ such that

$$\{(t, x(t), x'(t)) : t \in (\tau, \tau + \epsilon)\} \subset int(W).$$

We denote by W^i and W^{si}, respectively, the sets of ingress and strict ingress points of W.

It is clear that, if uniqueness of the solutions holds, then the consequent mapping is a single valued function.

Now assume that X, Y are topological spaces and let F be an abstract set-valued mapping which maps the points of X to nonempty compact subsets of Y. Then F is upper-semicontinuous (usc) at a point x_0 of X, if for any open subset A of $F(x_0)$ there exists a neibghborhood U of x_0 such that the set $F(x)$ is a subset of A for all points x of U.

The following lemmas give sufficient conditions for the upper semi-continuity of the consequent mapping and some useful properties for a class of usc mappings. Notice that the consequent mapping C is included in this class (see, e.g., [6]).

Lemma 2.1. *If for any point P of $S_c(W)$ all functions in $X(P)$ egress strongly from W, then the consequent mapping C is usc at any point P and the image $C(P)$ is a continuum subset of the boundary of W.*

Lemma 2.2. *Let X, Y be metric spaces and let $F : X \to 2^Y$ be a usc set-valued mapping. If A is a continuum subset of X such that for every $x \in A$ the image $F(x)$ is a continuum, then the image $F(A) := \cup\{F(x) : x \in A\}$ is also a continuum subset of Y.*

3. THE MAIN RESULTS

This section is devoted to the main results of the paper. We shall denote by $\bar{z}$ the conjugate of the complex number z and by $Re[z]$ its real part. Also the "typical" inner product in the n-dimensional space will be denoted by $< .,. >$.

Assume that the open set Ω has the property that there is a real number $R>0$ such that

$$V := \cup\{V(t) : t \in I\} \subset I \times \Omega,$$

where for each $t \in I$ we have set

$$V(t) := \{(t,x,y) : |x| \leq R, \ \ y \in \mathbb{C}^n\}.$$

In the sequel a bar over a matrix will denote the matrix with elements the complex conjugates of the elements of the original matrix.

Theorem. *Consider equation* (1.1) *where the continuous function* $f : J \times \Omega \to \mathbb{C}^n$ *satisfies the following conditions:*

(F_1) *For any* $t \in I$ *and* (t,x,y) *in the boundary of* $V(t)$ *the implication*

$$\text{if } Re[<\bar{x}, y>] = 0, \ \ \text{then } Re[<\bar{x}, f(t,x,y)> + |y|^2] \neq 0$$

holds.

(F_2) *There is a positive real number* M *such that any solution* $x \in X(V(0))$ *with* $|x'(0)| \leq M$, *satisfies the inequality*

$$|x'(t)| < M,$$

for all $t \in I_x$ *such that* $t > 0$ *and* $(t, x(t), x'(t)) \in V$.

Also assume that the nonsingular $n \times n$ *complex matrices* Q_0, Q_1 *are such that*

(M_1) *condition (1.5) is satisfied, and*

(M_2) *for all* $x, y \in \mathbb{C}^n$ *with*

$$Re[<\bar{x}, y>] \geq 0,$$

it holds

$$Re[<\bar{Q}_0^{-1}\bar{x}, Q_1^{-1}y>] > 0,$$

Then the problem (1.1), (1.2) *admits a solution* $x(t), t \in I$ *such that*

$$|x(t)| \leq R,$$

for all $t \in I$.

Proof. First of all we would like to notice some remarks:

(a) If we restrict the function f on a compact subset Z of $J \times \Omega$ containing the set V in its interior, we can approximate it uniformly on the set Z by a sequence of functions $f_k(t,x,y)$, which are at least C^1 on Z. For such functions we have uniqueness of solutions passing through points at least of the interior of Z. So, if we show the existence of a sequence of solutions (x_k) of the corresponding problems, with initial conditions in $V(0)$, then these solutions are uniformly bounded by R, their first derivatives by M and their second derivatives by the real number $sup\{|f(t,x,y)| : (t,x,y) \in Z\}$. Hence, by the Arzela-Ascoli's theorem a limiting point of this sequence exists which (according to continuous dependence arguments) will be a solution of the original problem.

(b) Let K be a compact subset of $\mathbb{C}^n \times \mathbb{C}^n$ containing the set

$$E := \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : |x| \leq R, \ \ |y| \leq M\}.$$

Define the continuous real valued function

$$S : (\lambda, x, y) \to Re[< e^{-i\lambda} \bar{Q}_0^{-1} \bar{x}, Q_1^{-1} y >]$$

and observe that, because of (M_2), there is a $\delta > 0$ such that

$$S(\lambda, x, y) > 0,$$

for all $(\lambda, x, y) \in [0, \delta] \times K$. Also, multiplying the matrix Q_0 by the complex factor $e^{i\lambda}$, for some real $\lambda \in (0, \delta)$, we can assume that the unit is not an eigenvalue of the matrix Q_0. Indeed, let us suppose that for each such λ, for which the matrix Q_0 does not have the unit as its eigenvalue, a solution x_λ exists for the corresponding problem. (Notice that each matrix of the form $e^{i\lambda} Q_0$ satisfies, also, condition (M_1).) Then, as in case (a) above, we can get an accumulation point (as the real parameter λ tends to zero), which by continuity, finally, will be a solution of the original problem.

Now consider the set W of all points (t, x, y) of V with $|y| \leq M$ and let W_0 be its cross section at $t = 0$, i.e. the set $W_0 := \{0\} \times E$. Let P_0 be a point in W_0 and let x be the unique solution passing from P_0. Then we distinguish two possibilities:

(i) The set I_x is a subset of I and it holds

$$|x(t)| < R,$$

for all $t \in [0, 1)$. Then we let $s := 1$. It is obvious that there is a point $P \in V(1)$ such that $(1, x(1), x'(1)) = P$.

(ii) Either

$$|x(0)| = R,$$

or there is a time $s \in I$ such that

$$|x(s)| = R \text{ and } |x(t)| < R,$$

for all $t \in [0, s)$.

In both these cases, from (F_2) we have

$$|x'(t)| < M,$$

for all $t \in (0, s)$.

We claim that the point $P := (s, x(s), x'(s))$ is a point of strict egress of the set W. Indeed, in case *(i)* this fact is obvious. So, consider case *(ii)*, where we can also assume that $s < 1$.

Define the real valued function

$$\phi(t) := |x(t)|^2 - R^2, \ \ t \in I_x,$$

and observe first that

$$\phi(s) = 0.$$

If

$$\phi'(s) = 2Re[< \bar{x}(s), x'(s) >] > 0,$$

then, clearly, $P \in W^{se}$, in case $s > 0$. If

$$\phi'(s) > 0, \text{ and } s = 0,$$

then

$$P = P_0 \in W^{se}.$$

Notice that in this case we have

$$|x(s)| = |x(0)| = |x_0| = R.$$

Then we can set

$$C(P) = P_0,$$

because the point P might be considered as the consequent point of itself.

If

$$\phi'(s) < 0,$$

then P is a point of strict ingress of W. Clearly, this fact cannot be true in case $s > 0$. If

$$s = 0 \text{ and } \phi'(0) < 0,$$

then the solution x must satisfy either (i), or (ii) above (for a certain new time $s > 0$).

These arguments lead us to discuss only the case

$$s > 0 \text{ and } \phi'(s) = 0.$$

The later means that

$$Re[< \bar{x}(s), x'(s) >] = 0$$

and, so, from (F_1)

$$\phi''(s) = 2Re[< \bar{x}(s), f(s, x(s), x'(s)) > + |x'(s)|^2] \neq 0.$$

The case

$$\phi''(s) < 0$$

is impossible. If

$$\phi''(s) > 0,$$

then we have

$$\phi(t) > 0, \text{ for all } t \in (s, s + \epsilon),$$

for some $\epsilon > 0$. Thus $P \in W^{se}$. Therefore our claim is true.

So far we have proved that

$$C(P_0) = P = (s, x(s), x'(s)).$$

From (1.5) we get

$$|Q_0 x(s)| \leq \|Q_0\| |x(s)| = 1.R = R \tag{3.1}$$

and

$$|Q_1 x'(s)| \leq \|Q_1\| |x'(s)| \leq 1.M = M. \tag{3.2}$$

Next, consider the set E as above and define the mappings

$$H : (x, y) \to (0, x, y) : E \to V(0) \text{ and } h : (t, x, y) \to (x, y) : V \to E,$$

as well as the matrix

$$Q := diag[Q_0, Q_1].$$

Then, from Lemmas 2.1, 2.2, our remark (a) (in the beginning of the proof) and relations (3.1), (3.2), we conclude that the function

$$T(x, y) := Qh(C(H(x, y))) : E \to E$$

maps continuously the closed, convex, bounded set E into itself. Hence, by the Brouwer's fixed point theorem it follows that there is a point $(x_0, y_0) \in E$ such that

$$T(x_0, y_0) = (x_0, y_0).$$

This means that there is a solution x such that $(x(0), x'(0)) = (x_0, y_0) \in E$ and

$$Q_0 x(s) = x_0 \text{ and } Q_1 x'(s) = y_0, \tag{3.3}$$

for some $s \in [0, 1]$.

To finish the proof, it is enough to show that (3.3) is true only for $s = 1$. Indeed, to prove it, we assume, on the contrary, that $s \in [0, 1)$. If $s = 0$, then, as we noticed above, $C(P) = P_0$ and so, it holds $x(s) = x_0$ and $x'(s) = y_0$. Hence from (3.3) we get $Q_0 x_0 = x_0$, where, notice that $|x_0| = R > 0$. This is impossible, because, from our remark (b) above, the unit is not a eigenvalue of the matrix Q_0.

Let us assume that $s \in (0, 1)$. We distinguish two cases:

Case A. Suppose that

$$|x_0| = |x(s)| = R.$$

Then, by the definition of the consequent mapping, the initial point P_0 must be an ingress point of W, so

$$\phi'(0) = 2Re[< \bar{x}(0), x'(0) >] = 2Re[< \bar{x}_0, y_0 >] \leq 0. \tag{3.4}$$

For the same reason the consequent point P is a point of egress of W, hence

$$\phi'(s) = 2Re[< \bar{x}(s), x'(s) >] \geq 0.$$

Then from (3.3) we derive

$$Re[< \bar{Q}_0^{-1}\bar{x}_0), (Q_1^{-1}y_0) >] = Re[< \bar{x}(s), x'(s) >] \geq 0.$$

This fact together with (3.4) contradict to (F_2).

Case B. Suppose that

$$|x_0| < R = |x(s)|.$$

Then we get

$$R = |x(s)| = |Q_0^{-1}x_0| \leq \|Q_0^{-1}\||x_0| < 1.R = R,$$

a contradiction. This completes the proof of the theorem. □

Remark. Hypothesis (F_2) holds, if, for instance, we impose a Nagumo type condition to the function $f(t,x,y)$, namely, if we assume that $f(t,x,y)$ has at most a quadratic growth rate in the argument y.

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